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Technical Notes

Mixed Formulation for Sensitivity Analysis of Laminated Conical Shells

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Nomenclature

A	=	membrane rigidity
B	=	coupling rigidity
D	=	flexural rigidity
G	=	geometry matrix
K	=	stiffness matrix
M	=	postbuckling matrix
M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$	=	moment resultants
$\hat{M}_{xx}$	=	bending moment applied at boundaries
N_{xx}, $N_{\theta\theta}$, $N_{x\theta}$	=	force resultants
$\hat{N}_{xx}$, $\hat{N}_{x\theta}$	=	axial and torsional forces applied at boundaries
$\bar{\mathbf{Q}}$	=	laminate transformed reduced stiffness matrix.
$\hat{Q}$	=	shearing force applied at boundaries
q_u, q_v, q_w	=	external distributed loading in axial, circumferential, and normal directions, respectively.
$r(x)$	=	radius at x coordinate; $x \sin\alpha$
$\mathbf{z}^{(0)}$	=	unknown vectors consisting of u, v, w, w_x, N_{xx}, $N_{x\theta}$, Q, and M_{xx} for prebuckling state
$\mathbf{z}^{(1)}$	=	unknown vectors consisting of u, v, w, w_x, N_{xx}, $N_{x\theta}$, Q, and M_{xx} for buckling state
$\mathbf{z}^{(2)}$	=	unknown vectors consisting of u, v, w, w_x, N_{xx}, $N_{x\theta}$, Q, and M_{xx} for postbuckling state
λ	=	load parameter deviating from bifurcation buckling load (λ_c)
ξ	=	amplitude of buckling mode
$()_{,x}$	=	derivatives with respect to axial (x) coordinate
$()_{,\theta}$	=	derivatives with respect to circumferential (θ) coordinate
$()_{/x}$	=	covariant derivatives with respect to axial (x) coordinate, given in Eq. (4)

I. Introduction

SHELL structures belong to the thin-wall family, the members of which are very sensitive to imperfections. This sensitivity depends on the geometry of the shell, its mechanical properties, and the external loading. There are two main approaches to investigating shell structure behavior: 1) tracing the entire nonlinear equilibrium path, which is a complicated process and entails a heavy computational effort; or 2) parametric study and rating of the shell in terms of its sensitivity. This latter approach, originally suggested by Koiter [1], provides qualitative rather than quantitative information on the characteristic buckling behavior of the shell.

Most of the numerous research works on this subject are confined to one of two formulations: u-v-w (e.g., for conical shells: [2] for the isotropic case; and [3] for the laminated case) and w-F, which is restricted to Donnell's theory [4] (e.g., for conical shells: [5,6] for the anisotropic case).

Unlike the isotropic case, the thickness and the material properties of laminated conical shells vary with the shell coordinates, ultimately resulting in coordinate-dependent stiffness matrices (**A**, **B**, and **D**). This effect complicates the analysis considerably, first due to the need to find an analytical representation for these functions (e.g., [7] for filament-winding process), and then by the solution of the set of nonlinear governing partial differential equations with variable coefficients. To avoid these obstacles, most investigators assumed constant stiffness coefficients. Based on the model developed by Baruch et al. [7], Goldfeld and Arbocz [8,9] calculated the buckling load of laminated conical shells, taking into consideration the variation in material properties with the shell coordinates. Goldfeld [3] also calculated the imperfection sensitivity of the same shell. All of the aforementioned researchers used a displacement-based formulation, and so their analytical models required the derivatives of the stiffness matrices in the axial direction. For complicated woven-fabric procedures, however, the derivative functions of the fiber orientations are not available or the constitutive functions have discontinuities. To overcome this problem, this study proposes using a mixed formulation, for which the main advantage is direct involvement of the stiffness matrices, without their derivatives.

The mixed-formulation approach consists of choosing a set of eight unknown functions as the boundary conditions obtainable from the variational formulation, namely, the displacements and rotation u, v, w, and w_x; and the resultant forces and moment N_{xx}, $N_{x\theta}$, Q, and M_{xx}. Thus, both kinematic and natural boundary conditions are satisfied directly. Additional advantages of the approach are involvement of only linear and quadratic operators in the governing equations, with the first derivatives of the unknown functions in the axial direction being the highest. These qualities significantly simplify calculation of the buckling and initial postbuckling states, and of Koiter's sensitivity b parameter [1]. The downside of the method is that there are eight unknowns, resulting in an increase in the required computational resources.

To demonstrate the mixed formulation, the nonlinear equilibrium differential equations are derived based on Donnell's kinematic approach [4]. The displacements u, v, and w; the rotation w_x; and the resultant forces and moment N_{xx}, $N_{x\theta}$, Q, and M_{xx} are the unknown dependent variables. The asymptotic technique is used to convert the nonlinear equations into eight linear sets. These equations are then solved by expanding of the dependent variables in the circumferential direction using Fourier series and in the axial direction using a special half-station finite difference scheme. Finally, the Galerkin procedure is used to minimize the error due to the truncated form of the series. For the sake of completeness, some parts of the procedure will be briefly recapitulated.

This study demonstrates the simplicity of the numerical model used to calculate the linear bifurcation buckling behavior, the initial postbuckling state, and especially Koiter's b parameter [1]. The study presents the quality of the mixed formulation in the case of laminated shell structure. Note that the numerical model of the imperfection sensitivity, which uses the mixed formulation presented in the current study, addresses laminated conical shells, but the procedure can be extended readily to any shells of revolutions obtained according to other shell theories as well.

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II. Governing Equations

The nonlinear equilibrium equations and the appropriate boundary conditions are derived on the basis of the potential-energy approach with the following energy Π:

$$\Pi = \frac{1}{2}\oint_\theta \int_x [N_{xx}\bar{\varepsilon}_{xx} + N_{\theta\theta}\bar{\varepsilon}_{\theta\theta} + N_{x\theta}\bar{\gamma}_{x\theta} + M_{xx}\chi_{xx} + M_{\theta\theta}\chi_{\theta\theta} + 2M_{x\theta}\chi_{x\theta}]\,\mathrm{d}x\,\mathrm{d}\theta - \oint_\theta \int_x (q_u u + q_v v + q_w w)\,\mathrm{d}x\,\mathrm{d}\theta - \oint_\theta (\hat{N}_{xx}u + \hat{N}_{x\theta}v + \hat{Q}w + \hat{M}_{xx}w_{,x})\,\mathrm{d}\theta \Big|_{x=0}^{x=l} \tag{1}$$

To attain equilibrium, the potential energy must be stationary. In this study, the kinematic relations and the constitutive equations are taken from [3], and so the first variation $\delta\Pi$ yields the following nonlinear equilibrium equations:

$$N_{xx/x} + \frac{N_{x\theta,\theta}}{r(x)} - \frac{\sin(\alpha)}{r(x)}N_{\theta\theta} + q_u = 0 \tag{2a}$$

$$N_{x\theta/x} + \frac{N_{\theta\theta,\theta}}{r(x)} + \frac{\sin(\alpha)}{r(x)}N_{x\theta} + q_v = 0 \tag{2b}$$

$$Q_{/x} + \frac{M_{\theta\theta,\theta\theta}}{r(x)^2} - \frac{N_{\theta\theta}}{r(x)}\cos(\alpha) + \frac{2\sin(\alpha)M_{x\theta,\theta}}{r(x)^2} + \frac{1}{r(x)}\left[N_{x\theta}w_{,x} + \frac{N_{\theta\theta}w_{,\theta}}{r(x)}\right]_{,\theta} + q_w = 0 \tag{2c}$$

$$-Q + M_{xx/x} + \frac{2M_{x\theta,\theta}}{r(x)} - \frac{\sin(\alpha)}{r(x)}M_{\theta\theta} + N_{xx}w_{,x} + \frac{N_{x\theta}w_{,\theta}}{r(x)} = 0 \tag{2d}$$

with the corresponding boundary conditions:

$$N_{xx} = \hat{N}_{xx} \quad \text{or} \quad u = \hat{u} \tag{3a}$$

$$N_{x\theta} = \hat{N}_{x\theta} \quad \text{or} \quad v = \hat{v} \tag{3b}$$

$$Q = \hat{Q} \quad \text{or} \quad w = \hat{w} \tag{3c}$$

$$M_{xx} = \hat{M}_{xx} \quad \text{or} \quad w_{,x} = \hat{w}_{,x} \tag{3d}$$

The covariant derivative $S_{/x}$ of an unknown function S, with respect to the contravariant variable (x) in Eqs. (2), is defined for a conical shell as

$$S_{/x} = S_{,x} + \frac{S}{r(x)}\sin\alpha \tag{4}$$

Accordingly, the mixed formulation involves a set of eight unknown functions: the displacements $u(x,\theta)$, $v(x,\theta)$, and $w(x,\theta)$; the rotation $w_{,x}(x,\theta)$; and the resultant forces and moment $N_{xx}(x,\theta)$, $N_{x\theta}(x,\theta)$, $Q(x,\theta)$, and $M_{xx}(x,\theta)$. Given are the four equilibrium equations: Eqs. (2); the constitutive equations for $N_{xx}(x,\theta)$, $N_{x\theta}(x,\theta)$ and $M_{xx}(x,\theta)$

$$N_{xx}(x,\theta) = A_{11}(x)\varepsilon_{xx} + A_{12}(x)\varepsilon_{\theta\theta} + A_{13}(x)\gamma_{x\theta} + B_{11}(x)\chi_{xx} + B_{12}(x)\chi_{\theta\theta} + B_{13}(x)\chi_{x\theta} \tag{5}$$

$$N_{x\theta}(x,\theta) = A_{31}(x)\varepsilon_{xx} + A_{32}(x)\varepsilon_{\theta\theta} + A_{33}(x)\gamma_{x\theta} + B_{31}(x)\chi_{xx} + B_{32}(x)\chi_{\theta\theta} + B_{33}(x)\chi_{x\theta} \tag{6}$$

$$M_{xx}(x,\theta) = B_{11}(x)\varepsilon_{xx} + B_{12}(x)\varepsilon_{\theta\theta} + B_{13}(x)\gamma_{x\theta} + D_{11}(x)\chi_{xx} + D_{12}(x)\chi_{\theta\theta} + D_{13}(x)\chi_{x\theta} \tag{7}$$

and the derivative of w in the axial direction (depending on the chosen numerical solution procedure).

Substitution of the constitutive equations and the kinematic relations into the governing equations leads to a set of nonlinear partial differential equations with variable coefficients, which can be written in operator form (see [10]). The operator coefficients are functions of the constitutive functions of $A_{ij}(x)$, $B_{ij}(x)$, and $D_{ij}(x)$, and of the cone geometry [radius $r(x)$ and cone semivertex angle α], and are therefore strong functions of the axial coordinate x.

Significant advantages of the mixed formulation are the exclusive involvement of linear and quadratic operators and the fact that the first derivative of the unknown function in the axial direction is the highest. Another important quality of this formulation lies in its direct satisfaction of both the natural $(N_{xx}, N_{x\theta}, Q, M_{xx})$ and essential (u, v, w, w_x) boundary conditions.

III. Imperfection Sensitivity Analysis

According to Koiter's general theory of elastic stability [1], the imperfection sensitivity of a structure is closely related to its initial postbuckling behavior, and the theory is exact in the asymptotic sense.

The classical buckling load of a perfect structure is denoted by λ_c, which is the load at which a nonaxisymmetric bifurcation from the prebuckling state occurs. Assuming that the eigenvalue problem for the buckling load λ_c yields a unique buckling mode $\mathbf{u}$, a solution that is valid in the initial postbuckling regime is sought in the form of the following asymptotic expansion:

$$\frac{\lambda}{\lambda_c} = 1 + a\xi + b\xi^2 + \ldots \tag{8}$$

$$\{S\} = \{S^{(0)}\} + \xi\{S^{(1)}\} + \xi^2\{S^{(2)}\} + \ldots \tag{9}$$

where S represents the unknown functions u, v, w, $w_{,x}$, N_{xx}, $N_{x\theta}$, Q, and M_{xx}; and the superscripts (0), (1), and (2) denote the prebuckling, buckling, and postbuckling states, respectively.

Formal substitution of this expansion in the nonlinear governing equations generates a sequence of equations for the functions appearing in the expansions, whereby the operators used are taken from [10].

The zeroth-order terms yield the partial differential equations of the prebuckling state:

$$L^{[e]}(S^{(o)}) + LL^{[e]}(S^{(o)}, T^{(o)}) = P^{[e]} \qquad e = 1\ldots 8 \tag{10}$$

The first-order terms yield those of the buckling state:

$$L^{[e]}(S^{(1)}) + [LL^{[e]}(S^{(o)}, T^{(1)}) + LL^{[e]}(S^{(1)}, T^{(o)})] = 0 \qquad e = 1\ldots 8 \tag{11}$$

and the second-order terms yield the partial differential equations of the postbuckling state:

$$L^{[e]}(S^{(2)}) + LL^{[e]}(S^{(o)}, T^{(2)}) + LL^{[e]}(S^{(2)}, T^{(o)}) = LL^{[e]}(S^{(1)}, T^{(1)}) \qquad e = 1\ldots 8 \tag{12}$$

where e denotes the equation number (eight equilibrium equations). The nonlinear solution of the prebuckling state [Eq. (10)] is obtained using a suitable numerical procedure that ultimately yields a limit point. To simplify the problem, the prebuckling state is solved by ignoring the nonlinear terms and solving only the linear part of Eq. (10), namely, $L^{[e]}(S^{(o)}) = p^{[e]}$. The applied loading $p^{[e]}$ consists of axial compression, internal or external pressure, and clockwise or counterclockwise torque. It is assumed to have a uniform spatial distribution and is divided into a fixed part and a variable part. The magnitude of the variable part is allowed to vary in proportion to a load parameter λ. This leads to an eigenvalue problem for the critical load λ_c. Neglecting the nonlinear deformation, which essentially uses the linear prebuckling solution, the equations governing the buckling state [Eq. (11)] become

$$L^{[e]}(S^{(1)}) + \lambda_c[LL^{[e]}(S^{(o)}, T^{(1)}) + LL^{[e]}(S^{(1)}, T^{(o)})] = 0 \qquad e = 1 \ldots 8 \tag{13}$$

where λ_c corresponds to the bifurcation buckling load.

IV. Numerical Solution Procedure

The solution of each set of equations admits separable solutions and is given in [10] as

$$S(x, \theta) = \sum_{m=o}^{2N_S} S_m(x) g_m(\theta) \tag{14}$$

where $2N_S$ is the number of retained terms in the relevant truncated Fourier series, and

$$g_m(\theta) = \begin{cases} \cos kjm\theta & m = 0, 1, 2, 3, \ldots N_S \\ \sin kjm\theta & m = N_S + 1, \ldots 2N_S \end{cases} \tag{15}$$

where j denotes the characteristic circumferential wave number; $k = 1$ is for the prebuckling and buckling states; $k = 2$ is for the postbuckling state; and $N_S = N_u$, N_v, N_w, $N_{w,x}$, N_{Nxx}, $N_{Nx\theta}$, N_Q, or N_{Mxx} according to the equation number. The θ dependence is eliminated by applying Galerkin's procedure. To reduce the ordinary differential equations to algebraic equations, the obtainable differential equations in the axial direction are solved using a special half-station finite difference scheme [11,12].

Thus, the following set of algebraic equation is obtained:

For the prebuckling state,

$$[\mathbf{K}]\{\mathbf{z}^{(0)}\} = \{\mathbf{p}\} \tag{16}$$

for the buckling state,

$$\{[\mathbf{K}] + \lambda[\mathbf{G}]\}\{\mathbf{z}^{(1)}\} = 0 \tag{17}$$

and for the postbuckling state,

$$[\mathbf{M}]\{\mathbf{z}^{(2)}\} = \{\mathbf{f}\} \tag{18}$$

Equation (17) is an eigenvalue problem in which λ represents the buckling load parameters and $\mathbf{z}^{(1)}$ denotes the buckling mode.

For the case of a membrane prebuckling state, Budiansky and Hutchinson [13] derived the well-known formula for Koiter's imperfection sensitivity b parameter [1]:

$$b = -\frac{\sigma_2 \cdot L_2(u_1) + 2\sigma_1 \cdot L_{11}(u_1, u_2)}{\lambda_c \sigma_0 \cdot L_2(u_1)} \tag{19}$$

The main goal in buckling analysis is prediction of the real buckling load. This can be done by using Koiter's theory to calculate the classical buckling load and the imperfection sensitivity parameter [1]. The buckling load for an imperfect structure occurs at the limit point λ_s, which is obtained by calculating the first derivative of λ with respect to ξ. For the case of linear prebuckling and assuming axisymmetric deformation ($a = 0$),

$$\left(1 - \frac{\lambda_s}{\lambda_c}\right)^{3/2} = \frac{3\sqrt{3}}{2} \xi \sqrt{-b} \left(\frac{\lambda_s}{\lambda_c}\right) \tag{20}$$

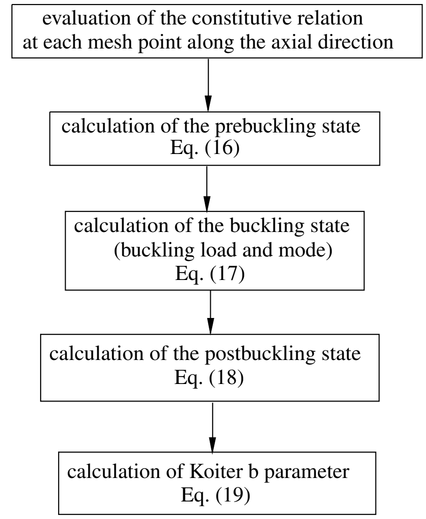

Fig. 1 Evaluation of the buckling state and Koiter sensitivity *b* parameter [1].

For a conical shell with variables u, v, w, $w_{,x}$, N_{xx}, $N_{x\theta}$, Q, and M_{xx}, the operators will be

$$\sigma_i \cdot L_{11}(u_j, u_k) = \int_a^b \int_0^{2\pi} \left\{ N_{xx}^{(i)} \left[w_{,x}^{(j)} w_{,x}^{(k)} \right] + N_{\theta\theta}^{(i)} \left[\frac{w_{,\theta}^{(j)} w_{,\theta}^{(k)}}{r(x)^2} \right] + 2N_{x\theta}^{(i)} \left[\frac{w_{,x}^{(j)} w_{,\theta}^{(k)}}{2r(x)} + \frac{w_{,x}^{(k)} w_{,\theta}^{(j)}}{2r(x)} \right] \right\} \mathrm{d}\theta \, \mathrm{d}x \qquad i, j, k = 0, 1, 2 \tag{21}$$

Here, the superscripts (i), (j), and (k) denote the appropriate state (prebuckling: 0, buckling: 1, and initial postbuckling: 2). Note that $L_{11}(u_j, u_j) = L_2(u_j)$ and $L_{11}(u_j, u_k) = L_{11}(u_k, u_j)$, and that a and b are, respectively, the initial and final coordinates along the axial direction ($[a, b] = [0, l]$ if the axial direction is measured along the cone surface).

Using the mixed formulation, $N_{xx}^{(i)}$ and $N_{x\theta}^{(i)}$ are derived directly for all three states ($i = 0, 1$, and 2). $N_{\theta\theta}^{(i)}$ is obtainable either from its constitutive law or, more simply, by extracting it from the first equilibrium equation [Eq. (2a)] and arriving at the following linear equation:

$$N_{\theta\theta}^{(i)} = \frac{r(x)}{\sin(\alpha)} N_{xx/x}^{(i)} + \frac{1}{\sin(\alpha)} N_{x\theta,\theta}^{(i)} \tag{22}$$

For the case of cylindrical shells ($\alpha = 0$), $N_{\theta\theta}^{(i)}$ is extracted from the second equilibrium equation [Eq. (2b)]:

$$N_{\theta\theta}^{(i)} = -\int_0^{2\pi} r(x) N_{x\theta,x}^{(i)} \, \mathrm{d}\theta \tag{23}$$

V. Conclusions

The present study suggests using a mixed formulation to simplify the calculation of initial postbuckling behavior and Koiter's b parameter for laminated shells [1]. This method's advantages over the commonly used u-v-w or w-F formulations are as follows: 1) direct involvement of the stiffness matrices, without their derivatives; 2) exclusive linear and quadratic operators in the governing equations; 3) the first derivatives in the axial direction of the unknown functions the highest; 4) direct compliance with both the natural and kinematic boundary conditions; and 5) direct evaluation of the Koiter's sensitivity b parameter [1].

These features make for a much simpler and more accurate analysis of the shell's behavior, which can be applied to any shells of revolution for the analysis of both initial postbuckling and full nonlinear behavior. The algorithm in Fig. 1 summarizes the implementation of the preceding method.

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